Vibration of Orthogonally Stiffened Waffle Cylinders Subjected to Initial Forces

Mirza I. Baig* and T. Y. Yang†
Purdue University, West Lafayette, Ind.

The kinetic energy expression is formulated for the orthogonally stiffened waffle cylindrical shell. The equations of motion are formulated by using Hamilton's principle for simply supported edge conditions. The effect of initial middle-surface stresses is included. A variety of examples are performed, and results are compared, whenever possible, with alternative solutions. It is found that, when the waffle cylinder is under uniform axial compression, the relation between the square of the frequency and the axial load is linear for each mode shape. This relation, however, becomes nonlinear when the cylinder is subjected to bending load.

Nomenclature

a = space between the stiffeners as shown in Fig. 1
d = depth of the stiffeners as shown in Fig. 1
E = modulus of elasticity
G = $E/2(1+\nu)$, shear modulus
h = thickness of the shell skin as shown in Fig. 1
L = length of the cylindrical shell as shown in Fig. 1
m,n = longitudinal half-wave and circumferential full-wave numbers
N_b = amplitude of linearly varying axial load (bending load)
N_c = uniform axial load
$\bar{N}_b$ = $N_b(1-\nu^2)/Eh$, nondimensionalized N_b
$\bar{N}_c$ = $N_c(1-\nu^2)/Eh$, nondimensionalized uniform load N_c
r = mean radius of the skin of the cylindrical shell
t_1,t_2 = width of the longitudinal and circumferential stiffeners, respectively
u,v,w = displacements in x, θ, and z directions, respectively
U_{mn}, V_{mn}, W_{mn} = displacement amplitudes
x,θ,z = cylindrical coordinates
$\epsilon_1,\epsilon_2,\epsilon_{12}$ = direct and shearing strain components
$\sigma_1,\sigma_2,\sigma_{12}$ = direct and shearing stress components
ν = Poisson's ratio
ρ = mass density
$\xi_{mn},\eta_{mn},\zeta_{mn}$ = time-dependent displacement amplitudes
ω = frequency of vibration
ω_s = $\pi/h(G/\rho)^{1/2}$, frequency of first simple thickness-shear mode of an infinite plate of thickness h
Ω = ω/ω_s, nondimensionalized frequency as defined by Eq. (15)

Introduction

THE use of cylindrical shells as structural components in aerospace vehicles has grown considerably in recent years. Because of the minimum weight requirement, the skin of the cylinder has to be thin. The thin skins buckle easily when subjected to middle-surface compressive stresses. They also vibrate easily when subjected to aerodynamic or thermal forces. Thus, the combined study of the buckling and vibrational problems for thin cylindrical shells has attracted considerable attention in the aerospace industry (see, for example, Refs. 1 and 2).

Because of the advances in material processing and accurate fabrication, shells stiffened with integrally milled grid bars in the form of waffle have been used widely. Stiffeners can increase substantially the buckling strength of the cylinder with little increase in weight. Studies of the buckling behavior or vibrational behavior of stiffened waffle cylinders have been extensive (see, for example, Refs. 3-6). But so far, the two behaviors have been treated only separately. It thus becomes the object of this paper to perform such a combined study for waffle cylinders. As an initial attempt, orthogonally stiffened waffle cylinders with simply supported edge conditions are investigated. Uniform and linearly varying axial loadings are considered.

The expressions for the strain energy and the potential energy due to external loads given in Ref. 4 are adopted in this study. The kinetic energy expression is formulated, and the equations of motion are derived from Hamilton's principle.

Each of the three displacement functions is assumed as the summation of a sufficient number of products of amplitudes and double trigonometric functions which satisfy the simply supported boundary conditions. The equations of motion are obtained by substituting the displacement functions into the energy expressions and then applying Hamilton's principle. The equations of motion are in the form of $3\times k\times \ell$ simultaneous equations, where k and ℓ are, respectively, the number of axial half-waves and the number of circumferential full-waves retained in the displacement functions. The eigenvalue solutions for these equations give the frequencies.

For the case of free vibration with or without uniform axial stress, the $3\times k\times \ell$ equations contain $k\times \ell$ sets of uncoupled equations, three in each set. Each set of equations, in general, gives three frequencies: Ω_1,Ω_2, and Ω_3 $(\Omega_1<\Omega_2<\Omega_3)$. The lowest frequency is associated with the mode dominated by the radial (flexural) motion, and the next two are associated with the mode dominated by the axial and circumferential (membrane) motions.

For the case of free vibration with linearly varying axial stress, there are k sets of coupled equations, $3\times \ell$ in each set. Each set must be solved simultaneously to obtain the $3\times \ell$ frequencies, of which the first ℓ frequencies correspond to the modes dominated by the radial (flexural) motion.

The formulations are first verified through examples that have alternative solutions. The examples include the vibration studies of a cylinder with longitudinal stringers and a cylinder with circumferential rings, both under uniform axial tension.

Received Jan. 26, 1976; revision received April 8, 1976.
Index categories: Structural Dynamic Analysis; Structural Stability Analysis.

*Graduate Student, School of Aeronautics and Astronautics. Student Member AIAA.
†Professor, School of Aeronautics and Astronautics. Associate Fellow AIAA.

The formulations are first used to analyze the effect of uniform and linearly varying axial loads on waffle cylinders with orthogonal stiffeners. It is found that, for the case of uniform axial compression, the relation between the square of the frequency and the axial compression is linear for each mode. For the case of linearly varying axial load, the relation becomes nonlinear.

Basic Assumptions

The following assumptions underlie this study:

1) The shell skin is stiffened integrally by longitudinal stringers and circumferential rings either externally or internally.

2) The stiffeners are spaced closely and equally such that they are considered continuously distributed over the shell surface.

3) The shell bends into a mode such that the lengths of the longitudinal wave and circumferential half-wave are much greater than the spacing between two adjacent rings and stringers, respectively (i.e., $L/m >> a$ and $\pi r/n >> a$).

4) Stiffeners are thin and deep such that uniaxial stress condition prevails. Shearing stress in the direction of the width of the stiffeners is negligible.

5) Material is linearly elastic, and plane cross sections remain plane after bending.

Energy Expressions

The strain energy expression has been derived in Ref. 4 under the same basic assumptions. The total strain energy U consists of two parts: U_1, the strain energy of the shell skin, and U_2, the strain energy of the stiffeners; i.e.,

$$U = U_1 + U_2 \tag{1}$$

Similarly, the total kinetic energy V also consists of two parts: V_1, the kinetic energy of the shell skin, and V_2, the kinetic energy of the stiffeners; i.e.,

$$V = V_1 + V_2 \tag{2}$$

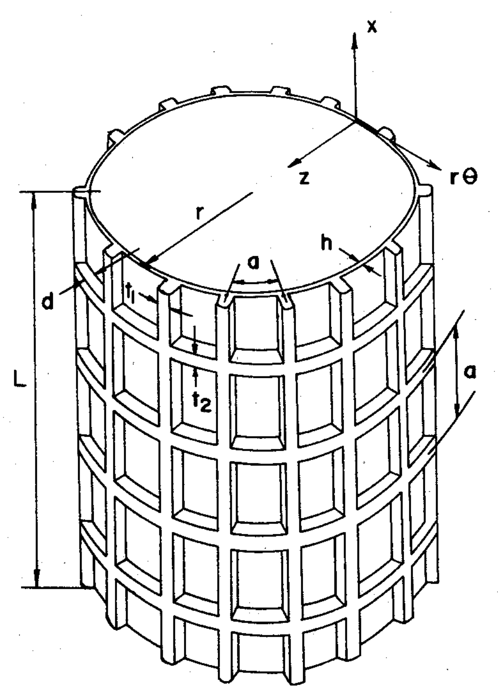

Fig. 1 An orthogonally stiffened waffle cylindrical shell with definitions of parameters.

For a circular cylindrical media, the kinetic energy is obtained as

$$V = \frac{1}{2}\int_0^{2\pi}\int_0^L\int \Big[(\dot{u} - z\dot{w}_x)^2 + \left\{\dot{v} - z\left(\frac{\dot{v}}{r} + \frac{\dot{w}_\theta}{r}\right)\right\}^2 + \dot{w}^2\Big] r\, dz dx d\theta \tag{3}$$

When the integration limits for dz in expression (3) are extended from $-h/2$ to $h/2$, the kinetic energy of the shell skin, V_1, is obtained; when the integration limits over the stiffener depth, the kinetic energy of the stiffeners, V_2, is obtained.

The kinetic energy for the shell skin is obtained from Eq. (3) as

$$V_1 = \frac{1}{2}\rho h \int_0^{2\pi}\int_0^L (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) r\, dx d\theta + \frac{1}{2}\rho\frac{h^3}{12}\int_0^{2\pi}\int_0^L \left(\dot{w}_x^2 + \frac{\dot{v}^2}{r^2} + \frac{2\dot{v}\dot{w}_\theta}{r^2} + \frac{\dot{w}_\theta^2}{r^2}\right) r\, dx d\theta \tag{4}$$

where the dot represents time derivative, and subscripts represent derivatives with respect to the indicated variables.

The kinetic energy for the stiffeners is obtained from Eq. (3) as

$$V_2 = \frac{1}{2}B\rho\Big[A\int_0^{2\pi}\int_0^L (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) r\, dx d\theta - AZ\int_0^{2\pi}\int_0^L 2\left(\dot{u}\dot{w}_x + \frac{\dot{v}^2}{r} + \frac{\dot{v}\dot{w}_\theta}{r}\right) r\, dx d\theta + AI\int_0^{2\pi}\int_0^L \left(\dot{w}_x^2 + \frac{\dot{v}^2}{r^2} + \frac{2\dot{v}\dot{w}_\theta}{r^2} + \frac{\dot{w}_\theta^2}{r^2}\right) r\, dx d\theta\Big] \tag{5}$$

The parameters A, I, and Z are the integral constants defined as

$$A = \int dz, \quad AZ = \int z\, dz, \quad AI = \int z^2\, dz \tag{6}$$

where the integrations are carried out through the depth of the internal or external stiffeners. The constant B is defined as

$$B = [(t_1 + t_2)a - t_1 t_2]/a^2 \tag{7}$$

where B is the ratio of the stiffener area to the shell skin area per (square) grid. Physically, this constant averages out the stiffener energy over the entire shell skin.

The potential energy of the applied middle-surface load is given in Ref. 4 as

$$W = \int_0^L\int_0^{2\pi} N_x\left(u_x + \frac{1}{2}w_x^2 + \frac{1}{2}v_x^2 + ew_{xx}\right) r\, dx d\theta \tag{8}$$

where N_x is the axial force per unit circumferential length, and e is the eccentric distance measured from the middle surface of the skin to the loading surface. The total energy of the waffle shell is thus

$$\Pi = U + V - W \tag{9}$$

Hamilton's principle states that over a period of time the total energy of a system remains constant; i.e.,

$$\delta\int_{t_1}^{t_2} \Pi\, dt = 0 \tag{10}$$

Application of this principle to the waffle shell system yields

$$\int_{t_1}^{t_2} (\delta U+\delta V-\delta W)dt=0 \tag{11}$$

The equations of motion are derived from the preceding principle.

Equations of Motion

For a cylindrical waffle shell with simply supported boundary conditions, the displacements may be represented as

$$u=\sum_{m=1}^{k}\sum_{n=0}^{\ell-1} \xi_{mn}\cos\frac{m\pi x}{L}\cos n\theta \tag{12a}$$

$$v=\sum_{m=1}^{k}\sum_{n=0}^{\ell-1} \eta_{mn}\sin\frac{m\pi x}{L}\sin n\theta \tag{12b}$$

$$w=\sum_{m=1}^{k}\sum_{n=0}^{\ell-1} \zeta_{mn}\sin\frac{m\pi x}{L}\cos n\theta \tag{12c}$$

and

$$\begin{aligned}\xi_{mn},\eta_{mn},\zeta_{mn}&=\xi_{mn}(t),\eta_{mn}(t),\zeta_{mn}(t)\\ &=U_{mn}\sin\omega t,\ V_{mn}\sin\omega t,\ W_{mn}\sin\omega t\end{aligned} \tag{13}$$

where ω is the circular frequency of vibration, and U_{mn}, V_{mn}, and W_{mn} are the amplitudes.

Substituion of the displacement functions (12) into the energy expressions (1, 2, 4, 5, 8, and 9) and then application of the criterion (11) yield a system of homogeneous equations

$$\left[\begin{bmatrix} C_{11} & C_{12} & C_{13}\\ C_{12}^T & C_{22} & C_{23}\\ C_{13}^T & C_{23}^T & C_{33}\end{bmatrix} - \bar{N}_x\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & F_{33}\end{bmatrix} - \Omega^2\begin{bmatrix} M_{11} & M_{12} & M_{13}\\ M_{12}^T & M_{22} & M_{23}\\ M_{13}^T & M_{23}^T & M_{33}\end{bmatrix}\right]\begin{Bmatrix}\xi_{mn}\\ \eta_{mn}\\ \zeta_{mn}\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \tag{14}$$

where the nondimensional parameters are defined as

$$\bar{N}_x=N_x(1-\nu^2)/Eh,\quad \Omega=\omega/\omega_s=\omega h/\pi(\rho/G)^{1/2} \tag{15}$$

The stiffness submatrices C_{ij} are given in Ref. 4. The mass submatrices for internally stiffened shells are derived as follows:

$$M_{11}(m,n)=\frac{\pi^2}{h^2}\cdot G\left\{h+\frac{(t_1+t_2)a-t_1t_2}{a^2}\cdot d\right\}(1+\delta_{0n}) \tag{16a}$$

$$M_2(m,n)=0 \tag{16b}$$

$$M_{13}(m,n)=-\frac{\pi^2}{h^2}\cdot G\left\{\frac{(t_1+t_2)a-t_1t_2}{a^2}\,\frac{d(d+h)}{2}\right\}$$
$$\times\frac{m\pi}{L}\times(1+\delta_{0n}) \tag{16c}$$

$$M_{22}(m,n)=\frac{\pi^2}{h^2}\cdot G\left[\left(h+\frac{h^3}{12r^2}\right)+\frac{(t_1+t_2)-t_1t_2}{a^2}\right.$$
$$\left.\times\left\{d-\frac{d(d+h)}{r}+\frac{d}{r^2}\left(\frac{d^2}{3}+\frac{hd}{2}+\frac{h^2}{4}\right)\right\}\right](1+\delta_{0n}) \tag{16d}$$

$$M_{23}(m,n)=\frac{\pi^2}{h^2}\cdot G\left[-\frac{h^3}{12r}+\frac{(t_1+t_2)a-t_1t_2}{a^2}\left\{\frac{d(d+h)}{2}\right.\right.$$
$$\left.\left.-\frac{d}{r}\left(\frac{d^2}{3}+\frac{hd}{2}+\frac{h^2}{4}\right)\right\}\right]\times\left(\frac{n}{r}\right)(1+\delta_{0n}) \tag{16e}$$

$$M_{33}(m,n)=\frac{\pi^2}{h^2}\cdot G\left[\left\{h+\frac{(t_1+t_2)a-t_1t_2}{a^2}\cdot d\right\}+\left\{\frac{h^3}{12}\right.\right.$$
$$\left.+\frac{(t_1+t_2)a-t_1t_2}{a^2}\cdot d\left(\frac{d^2}{3}+\frac{hd}{2}+\frac{h^2}{4}\right)\right\}$$
$$\left.\left\{\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{r}\right)^2\right\}\right](1+\delta_{0n}) \tag{16f}$$

where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$. Those for externally stiffened shells are

$$M_{11}(m,n)=\frac{\pi^2}{h^2}\cdot G\left\{h+\frac{(t_1+t_2)a-t_1t_2}{a^2}\cdot d\right\}(1+\delta_{0n}) \tag{16g}$$

$$M_{12}(m,n)=0 \tag{16h}$$

$$M_{13}(m,n)=\frac{\pi^2}{h^2}\cdot G\left\{\frac{(t_1+t_2)a-t_1t_2}{a^2}\cdot\frac{d(d+h)}{2}\right\}$$
$$\times\left(\frac{m\pi}{L}\right)(1+\delta_{0n}) \tag{16i}$$

$$M_{22}(m,n)=\frac{\pi^2}{h^2}\cdot G\left[\left(h+\frac{h^3}{12r^2}\right)+\frac{(t_1+t_2)a-t_1t_2}{a^2}\right.$$
$$\left.\left\{d+\frac{d(d+h)}{r}+\frac{d}{r^2}\left(\frac{d^2}{3}+\frac{hd}{2}+\frac{h^2}{4}\right)\right\}\right](1+\delta_{0n}) \tag{16j}$$

$$M_{23}(m,n)=\frac{\pi^2}{h^2}\cdot G\left[-\frac{h^3}{12r}+\frac{(t_1+t_2)a-t_1t_2}{a^2}\right.$$
$$\left.\left\{-\frac{d(d+h)}{2}-\frac{d}{r}\left(\frac{d^2}{3}+\frac{hd}{2}+\frac{h^2}{4}\right)\right\}\right]\times\left(\frac{n}{r}\right)(1+\delta_{0n}) \tag{16k}$$

$$M_{33}(m,n)=\frac{\pi^2}{h^2}\cdot G\left[\left\{h+\frac{(t_1+t_2)a-t_1t_2}{a^2}\cdot d\right\}+\left\{\frac{h^3}{12}\right.\right.$$
$$\left.+\frac{(t_1+t_2)a-t_1t_2}{a^2}\cdot d\left(\frac{d^2}{3}+\frac{hd}{2}+\frac{h^2}{4}\right)\right\}$$
$$\left.\left\{\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{r}\right)^2\right\}\right](1+\delta_{0n}) \tag{16l}$$

The coefficients of submatrix F_{33} are as follows: for uniform axial compression,

$$F_{33}(m,n)=[Eh/(1-\nu^2)]\cdot(m\pi/L)^2(1+\delta_{0n}) \tag{17a}$$

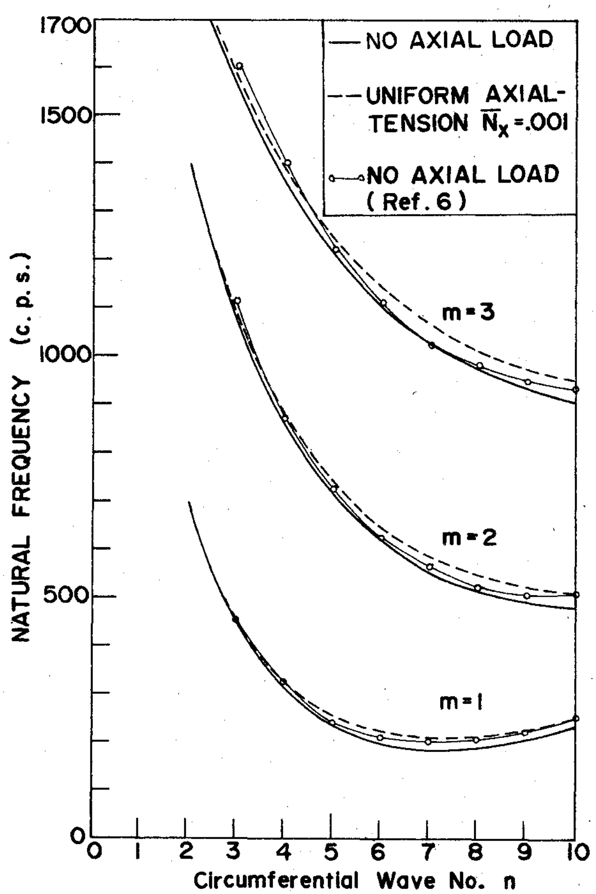

Fig. 2 Natural frequencies for a simply supported cylinder with external longitudinal stringers.

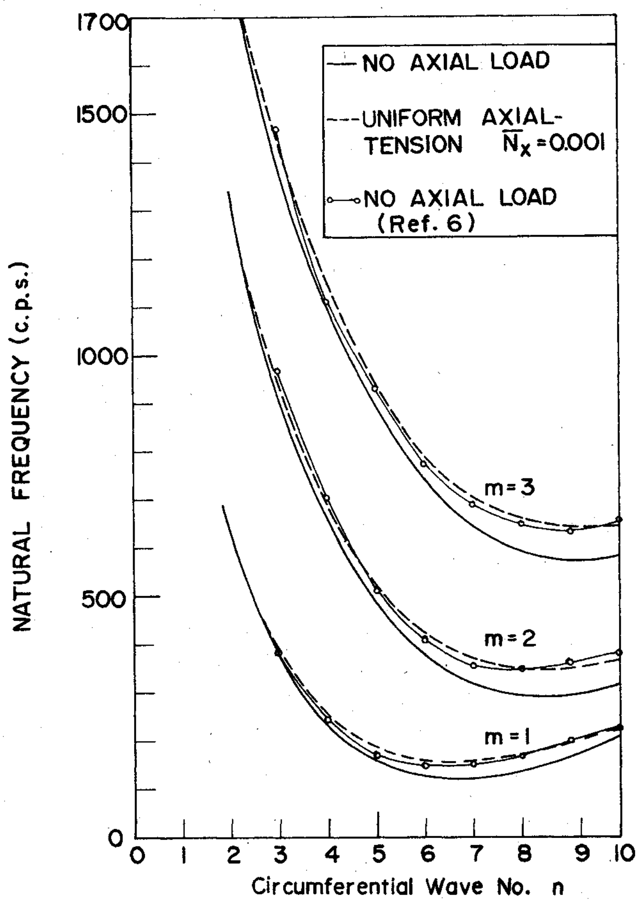

Fig. 3 Natural frequencies for a simply supported cylinder with internal longitudinal stringers.

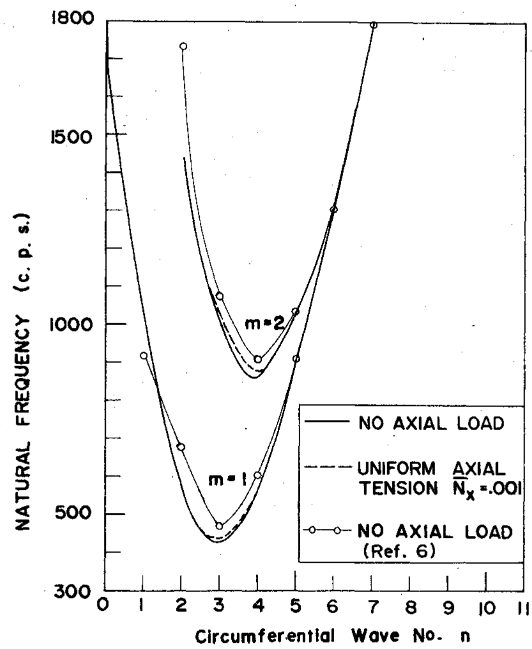

Fig. 4 Natural frequencies for a simply supported cylinder with external circumferential rings.

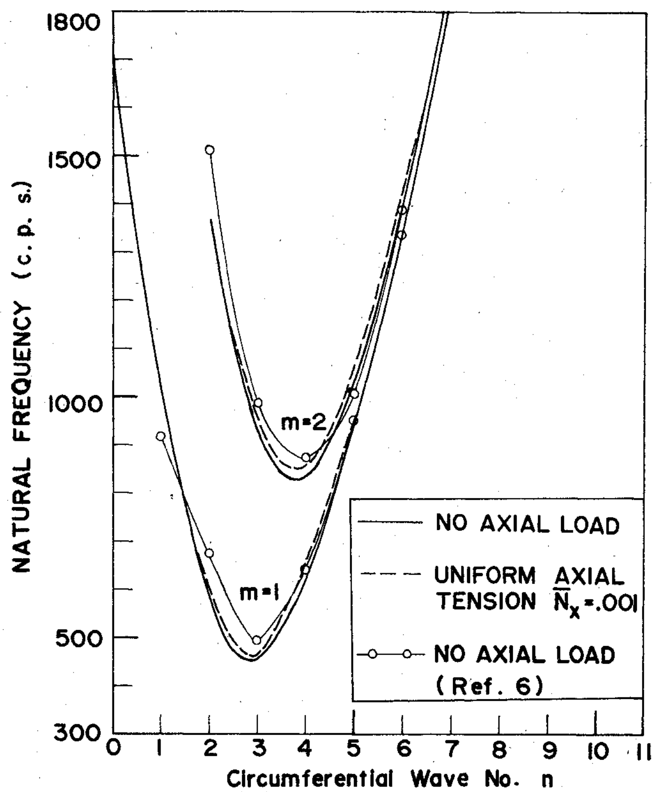

Fig. 5 Natural frequencies for a simply supported cylinder with internal circumferential rings.

and, for linearly varying axial load,

$$F_{33}(m,n) = [Eh/(1-\nu^2) \cdot \tfrac{1}{2}(m\pi/L)^2(1+\delta_{1n}-\delta_{0n}) \quad (17b)$$

Results

A. Simply Supported Cylinders with Longitudinal Stringers

A cylinder with external stringers was considered first. The cylinder was defined by the following parameters: $r=9.55$ in., $L=23.75$ in., $h=0.028$ in., $t_1=0.096$ in., $t_2=0$ in., $d=0.32$

in., $a=1$ in., $E=10.5\times10^6$ psi, $\nu=0.3$, and $\rho=2.6139\times10^{-4}$ lb-sec^2/in.^{4}. The natural frequencies for various longitudinal and circumferential wave numbers are plotted in Fig. 2. This example was analyzed previously by Mikulas and McElman.[6] In the kinetic energy expression given in Ref. 6, all of the rotatory inertias and translatory inertias in the middle surface were disregarded. All of those terms were retained in this study. The results for frequencies given in Ref. 6 also are shown in Fig. 2 for comparison, and it is seen that their frequencies are slightly higher. The effect of uniform axial tension also was studied. It is seen in Fig. 2 that such stiffening effect raised the frequencies.

The cylinder with internal stringers then was analyzed. The results are plotted in Fig. 3. Again, the frequencies from Ref. 6 are seen in Fig. 3 to be higher. The stiffening effect due to the uniform initial tension is seen clearly in the figure.

B. Simply Supported Cylinders with Circumferential Rings

A cylinder with external rings was considered first. The cylindrical shell was the same as that in case A. The cross section and the spacing of the rings are also the same as those of the stringers in case A. The frequencies for various mode numbers are shown in Fig. 4. The results from Ref. 6 also are shown. The frequencies from Ref. 6 are, in general, higher. Slight uniform tension was included in the analysis, and it is seen in Fig. 4 that the resulting stiffening effect is less than that shown in Fig. 2. The cylinder with internal rings then was analyzed. Results are shown in Fig. 5 which are in good agreement with those found in Ref. 6.

C. Orthogonally Stiffened Waffle Cylinder under Uniform Axial Load

A simply supported cylinder stiffened internally with both stringers and rings was considered. The parameters were assumed as $r=48$ in., $L=50$ in., $h=0.05$ in., $t_1=t_2=0.125$ in., $d=0.2$ in., $a=3$ in., $E=10.6\times10^6$ psi, $\nu=0.333$, and $\rho=2.6139\times10^{-4}$ lb-sec^2/in.^{4}. The results for frequencies for various mode numbers are shown in Fig. 6. The results obtained by neglecting the stiffeners also are shown in Fig. 6. It is seen in Fig. 6 that, for modes with higher longitudinal wave number, the frequencies are less influenced by the change of circumferential wave number. It also is seen that, for modes with lower longitudinal wave number, the differences in frequency between the waffle cylinder and monocoque cylinder are less.

The effect of uniform axial compression then was studied. The results are plotted as the square of nondimensional frequency (Ω^2) vs the uniform axial compression ($\bar{N}_c$) for various modes in Fig. 7. It is seen that, for each mode, the relation between Ω^2 and $\bar{N}_c$ is linear. The linear behavior has been reported previously for the case of monocoque cylinders.[1,2]

D. Orthogonally Stiffened Waffle Cylinder under Bending Load at the Edges

The linearly varying axial load (bending load) in the form that $N_x=N_b\cos\theta$ was considered. Since simply supported edge conditions were considered, the longitudinal component in each displacement series in Eq. (12) was represented by one term. Because the axial load was not distributed uniformly, the circumferential component in each displacement series was represented by the summation of 28 terms. The eigenvalue solution of a set of 84 equations was required.

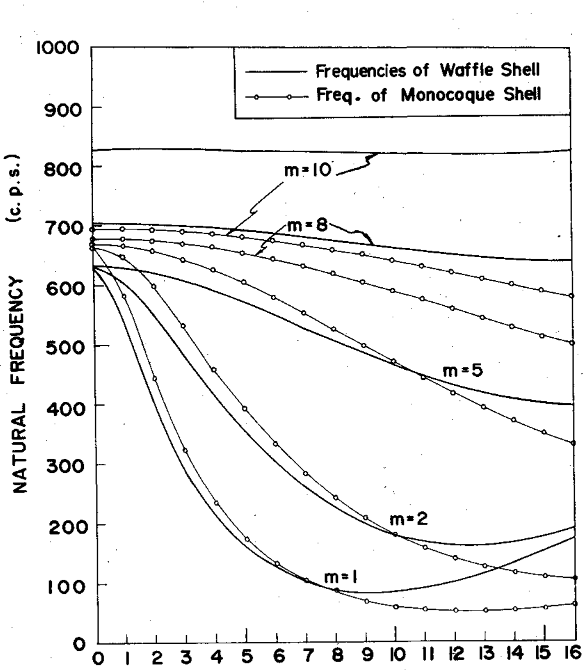

Fig. 6 Natural frequencies for a simply supported cylinder with internal stringers and rings.

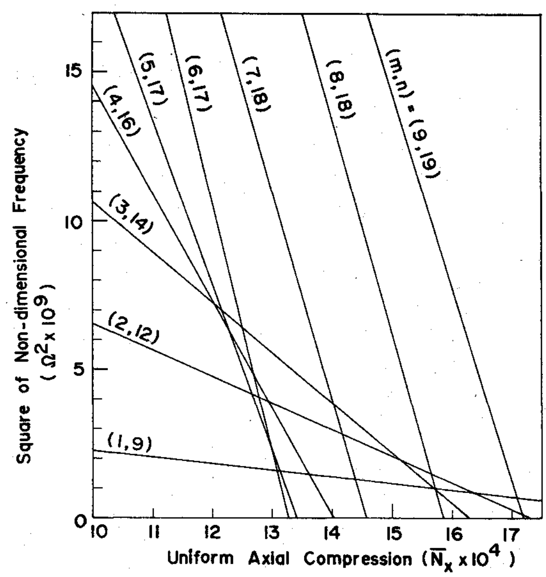

Fig. 7 Variation of the square of nondimensional frequency with the intensity of axial compression for a simply supported waffle cylinder for various modes.

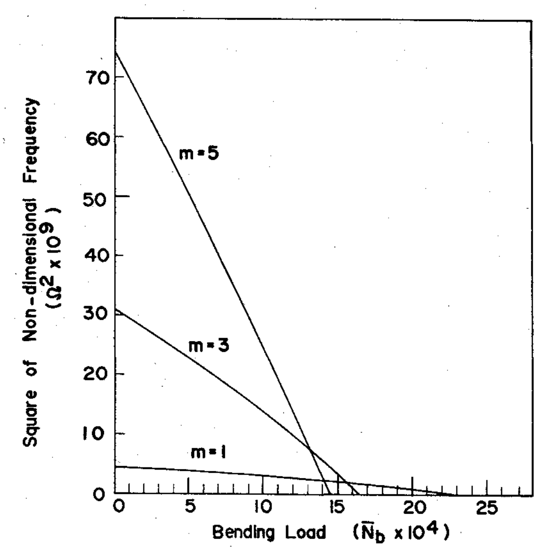

Fig. 8 Variation of the square of nondimensional frequency with the intensity of bending load for a simply supported waffle cylinder for various modes.

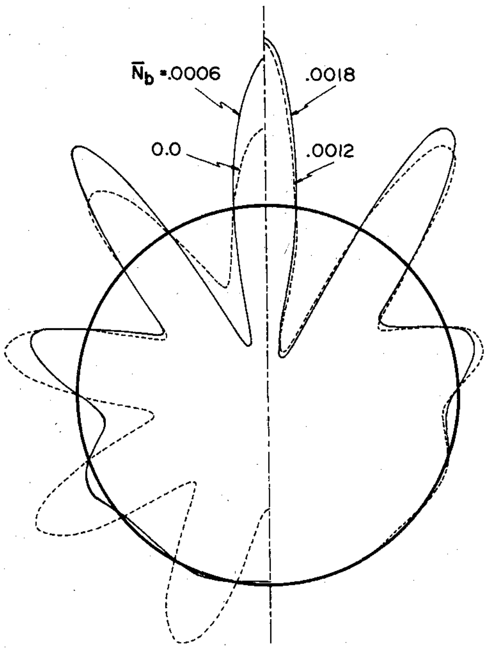

Fig. 9 Variation of the circumferential mode shape vs the bending load for the mode with $m = 1$.

The present results are shown in Fig. 8 as a plot of the square of nondimensional frequency vs the intensity of bending load (maximum axial stress) for three different mode shapes. It is interesting to see that such relations are no longer linear. Such nonlinear relations have been reported for the case of monocoque cylinders.[2] It also is seen in Fig. 8 that the three curves cross each other. This phenomenon indicates that for this example the frequency values (with no initial bending load) are lower for modes with lower longitudinal wave numbers, but the buckling load values (with no vibration) are higher for modes with lower longitudinal wave numbers.

Figure 9 shows the circumferential mode shapes corresponding to one-half longitudinal wave ($m=1$) for four different values of bending load: $\bar{N}_b = 0$, 0.006, 0.0012, and 0.0018, respectively. It is seen that the amplitudes of the waves increase in the compressive zone and decrease in the tensile zone as the bending load is increased, whereas the nodal locations of each circumferential wave practically remain unchanged.

Concluding Remarks

The kinetic energy expression for an orthogonally stiffened waffle cylinder has been derived. The expressions for the strain energy and potential energy due to external loads have been adopted from Ref. 4. The equations of motion have been derived by using Hamilton's principle.

The waffle cylinders with simply supported edge conditions under uniform and linearly varying axial loadings have been considered. The results have been presented by plotting the square of frequency vs the values of load for each mode shape. The relations have been found to be linear for the case of uniform axial load but nonlinear for the case of linearly varying load. The findings agree with previous results for the case of monocoque cylinders.[1,2] For the case of linearly varying load, the shifting of wave amplitudes from the tensile zone to the compressive zone is observed and presented.

References

[1] Armenakas, A. E., "Influence of Initial Stress on the Vibration of Simply Supported Circular Cylindrical Shells," *AIAA Journal*, Vol. 2, Sept. 1964, pp. 1607-1612.

[2] Yang, T. Y. and Kim, H. W., "Vibration and Buckling of Shells Under Initial Stresses," *AIAA Journal*, Vol. 11, Nov. 1973, pp. 1525-1531.

[3] Meyer, R. R., "Buckling of 45° Eccentrically Stiffened Waffle Cylinders," *Journal of the Royal Aeronautical Society*, Vol. 71, July 1967, pp. 516-520.

[4] Baig, M. I. and Yang, T. Y., "Buckling Analysis of Orthogonally Stiffened Waffle Cylinders," *Journal of Spacecraft and Rockets*, Vol. 111, Dec. 1974, pp. 832-837.

[5] Hoppmann, W. H., "Some Characteristics of the Flexural Vibrations of Orthogonally Stiffened Cylindrical Shells," *The Journal of the Acoustical Society of America*, Vol. 30, Jan. 1958, pp. 77-83.

[6] Mikulas, M. M. and McElman, J. A., "On Free Vibration of Eccentrically Stiffened Cylindrical Shells and Flat Plates," NASA-TN-D-3010, Sept. 1965.